

Relevance of symmetry methods in mechanics of materials

J.-F. Ganghoffer · V. Magenet · R. Rahouadj

Received: 10 June 2009 / Accepted: 29 June 2009 / Published online: 16 July 2009
© Springer Science+Business Media B.V. 2009

Abstract The interest and relevance of symmetry methods as a predictive and systematic methodology in the continuum mechanics of materials is analyzed, relying on a classification of the inherent aspects in terms of the direct, extended direct, and inverse methods. Although being interrelated, these three problems each have a specific argumentation which is separately exposed in the present contribution. The direct problem of finding invariants associated with a given constitutive law for materials, including dissipation, is first envisaged. The abstract formulation of constitutive laws in terms of the state laws and a dissipation potential expressing the evolution of internal state variables is considered, in the framework of irreversible thermodynamics. It is shown that a specific choice of the components of the symmetry vector acting in the space of independent and dependent variables leads to a local invariance condition of the constitutive law fully equivalent to the variational symmetry condition using the rate of the internal energy density. As a specific situation involving this methodology, a time–temperature equivalence principle of polymers is obtained from the requirement of group invariance of the field equations. A validation of this invariance principle is given by a comparison of the modelled master response and the master curve constructed from a set of experimental results at various temperatures. The extended direct method is next presented as a generalization of the direct method, in the sense that a classification of constitutive functions modelling the material behavior is achieved via a symmetry analysis. In the third part of the paper, the inverse problem of constructing a material's constitutive law exploiting a postulated Lie-group structure is exposed. A constitutive model is then identified which satisfies the symmetries exhibited by the experimental data.

Keywords Constitutive laws · Continuum mechanics of materials · Dissipative media · Lie symmetries · Master curves · Thermodynamics of relaxations · Williams–Landel–Ferry relation

J.-F. Ganghoffer · R. Rahouadj
LEMTA-ENSEM, 2, Avenue de la Forêt de Haye, B.P. 160, 54504 Vandoeuvre-lès-Nancy, Cedex, France

J.-F. Ganghoffer
e-mail: jean-francois.ganghoffer@ensem.inpl-nancy.fr

R. Rahouadj
e-mail: rachid.rahouadj@ensem.inpl-nancy.fr

V. Magenet (✉)
LAEGO-ENSG, Rue du Doyen Marcel Roubault, B.P. 40, 54501 Vandoeuvre-lès-Nancy, France
e-mail: vincent.magenet@ensg.inpl-nancy.fr

1 Introduction

The power of Lie-group symmetry analysis has been extensively used to date in a rather mathematical sense, essentially to support the finding of analytic solutions to partial differential equations (see [1–3] for a body or results). For a given DE problem, one can algorithmically calculate its admitted point symmetries—transformations of dependent and independent variables that map a problem into itself (e.g. [4–7]). Knowledge of admitted symmetries allows one to construct mappings relating DE systems, find out whether or not a given nonlinear DE system can be mapped into a linear system by an invertible transformation, and find exact (group-invariant or symmetry-generated) solutions. Lie-group analysis is of further interest in setting up numerical schemes that preserve the group properties of an initial boundary-value problem (BVP); see [8].

In the context of continuum solid mechanics, Lie groups have been applied to solve the Navier and the Lamé equations (see [9–11]), or, in a similar spirit and extending this view to dissipation, to partially solve the ideal plasticity equation ([12, 13]). The concept of nonlocal symmetries allows one to construct novel BVP in continuum mechanics (and group-invariant solutions), involving potential variables, thereby extending the classical picture relying on the traditional Lagrangian and Eulerian viewpoints, [14]. The field of Eshelbian Mechanics (in honour of the work of Eshelby), otherwise coined Configurational Mechanics, relies on translational symmetries in the so-called material space, for the writing of field equations in terms of Eshelby stresses. Those symmetries extended to rotations and dilatations have been intensively used to construct the well-known J-integrals; see [15–18].

In contrast to this, the possible interest of using Lie symmetries in the mechanics of materials has clearly not been explored so far to the same extent. The focus in this contribution is on the involvement of Lie groups as a new predictive and systematic methodology to obtain invariance properties of materials: more precisely, from the knowledge of the constitutive law of a given material, the ability of Lie symmetries to predict its response under various control conditions (this terminology will be defined later on) will be assessed. The interactions between those fields can be summarized in terms of the following three generic interrelated problems:

- The Direct Method (abbreviation DM) is exposed in detail in Sect. 2 and can be summarized as follows: for a given material and a given constitutive model, which can be written in a physical framework such as irreversible thermodynamics, find the symmetries induced by the postulated constitutive law. This methodology has been followed historically in the literature by several authors; see e.g. [1–3]. In the context of continuum solid mechanics, this approach may be especially promising for dissipative materials for which the constitutive law includes the state laws for the observable variables (such as stress and strain) and the internal state laws expressing the evolution of the internal variables. The invariants associated with those symmetries may then be calculated, and used in a practical way to synthesize the material's response into so-called master curves, revealing the variation of the material's response when some control parameters, such as temperature or strain rate, vary;
- The Extended Direct Method (abbreviated as EDM), also exposed in Sect. 2, can be stated as follows: as an extension of the direct problem, find and classify the symmetries in a given boundary-value problem involving some initially unspecified constitutive functions (and also possibly loading functions). This approach has been recently followed in nonlinear elasticity [14]; invariants can also be constructed from the obtained symmetries. Indeed, the use of nonlocal variables leads further to novel and more general formulations in Continuum Mechanics;
- The Inverse Method (abbreviated as IM) is presented in Sect. 3, and can be summarized as follows: relying on experimental data obtained from a set of adequate mechanical testings, use the postulated group structure of those data to construct a possible material's constitutive law via the Lie symmetry analysis, in terms of initially unknown constitutive functions. The Lie symmetries can here be viewed as an interpolation method allowing to continuously link experimental data, but relying on a limited, well chosen, set of experiments.

The inverse problem is clearly tied to the direct problem: once a possible material constitutive law has been constructed, the material's response can be predicted (outside the range of variation of the control variables involved in the measurements) for varying experimental conditions, hence using Lie symmetries inherent to the obtained constitutive law as an extrapolation technique. Observe that this methodology of exploiting Lie symmetries can

be potentially applied to a wide class of materials, including polymers, metals, ceramics, metallic and polymeric foams, and constitutive laws, such as nonlinear elasticity, viscoelasticity and viscoplasticity.

The following conventions will be adopted in the sequel: vectors and matrices are denoted by bold-face letters. The total derivative of a functional expression, e.g. $u(t, f(t))$, with respect to time t will be written as $\dot{u} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial f} \dot{f}$, accounting for both the explicit and the implicit dependence of the functional u upon time.

In order to clarify the presentation of the methodology at the heart of the present work, the exposition of ideas related to the three presented methods will be summarized in separate diagrams synthesizing the different logical steps of each method in terms of phases. The Direct, Inverse, and Extended Direct Methods will be conveniently abbreviated as DM, IM, and EDM, respectively; for a given approach, each phase will accordingly be designated by the letter P followed by a number indicating the order in the phase sequence, e.g. DM-P1 refers to the first phase of the Direct Method.

2 The direct method

The DM algorithm is summarized in the diagram of Fig. 1. The associated constitutive model accounted for in the present illustration is based on a thermodynamic approach (Phase DM-P1), a reminder of which is given next.

2.1 Construction of the constitutive law: case of TIP

In mechanics, constitutive laws for dissipative materials are generally written under the umbrella of the thermodynamics of irreversible processes (TIP), adopting one of its possible variants [19]. Relying on Callen’s axiomatic,

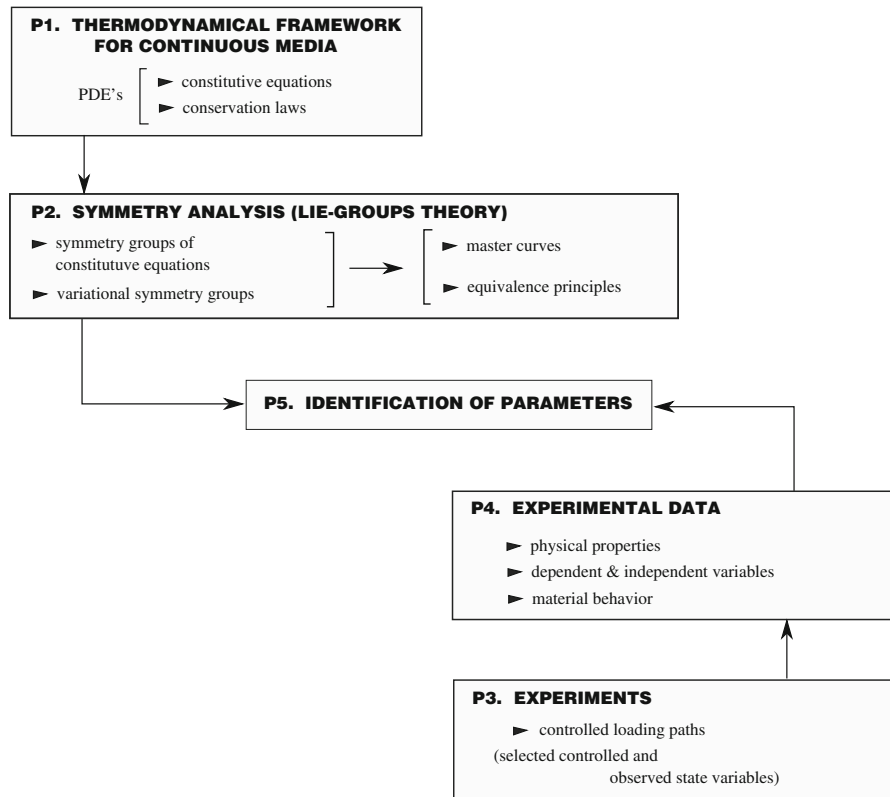


Fig. 1 Diagram associated with the Direct Method

[20], and a generalization of De Donder’s thermodynamics, let the internal potential energy $E(\mathbf{y}, \mathbf{z})$ characterizing the medium, which depends on a set of extensive variables $\mathbf{y} = (S, V\varepsilon, N_k \dots)$, where S , $V\varepsilon$, and N_k are, respectively, the entropy, the volume-weighted deformation, and the number of moles of the k th species. Those last variables are identified with a set of internal variables \mathbf{z} describing the irreversible evolution of the microstructure (such as plasticity, viscoplasticity, and damage). The thermodynamical system is a finite volume V with boundary ∂V of a solid continuum body. The extensity property of E is expressed as

$$E(\lambda \mathbf{y}, \lambda \mathbf{z}) = \lambda E(\mathbf{y}, \mathbf{z}). \tag{1}$$

Taking the derivative of the previous equation with respect to λ at $\lambda = 1$, one obtains the Euler relation in the form of the Gibbs relation

$$E(\mathbf{y}, \mathbf{z}) = \mathbf{Y}(\mathbf{y}, \mathbf{z}) \cdot \mathbf{y} - \mathbf{A}(\mathbf{y}, \mathbf{z}) \cdot \mathbf{z} \tag{2}$$

satisfied by the internal energy $E(\mathbf{y}, \mathbf{z})$, with $\mathbf{Y} = E_{,y}$ the intensive variables (vector) and $\mathbf{A} = -E_{,z}$ the generalized non-equilibrium forces, respectively. The intensive variables are dual with respect to the extensive ones in a thermodynamical sense. For example, the Cauchy stress, viz. the second-order tensor σ , is conjugated to the strain, viz. the second-order tensor ε . The extensive variables are control variables in the sense that they are the arguments of the appropriate thermodynamical potential. When the Gibbs relation is accounted for:

$$\frac{dE}{dt}(\mathbf{y}, \mathbf{z}) = \mathbf{Y}(\mathbf{y}, \mathbf{z}) \cdot \frac{d\mathbf{y}}{dt} - \mathbf{A}(\mathbf{y}, \mathbf{z}) \cdot \frac{d\mathbf{z}}{dt}, \tag{3}$$

the Gibbs–Duhem relation results from the differentiation of (2), viz.

$$\mathbf{y} \cdot \frac{d\mathbf{Y}}{dt} - \mathbf{z} \cdot \frac{d\mathbf{A}}{dt} = 0. \tag{4}$$

A total time derivation of the intensive variables \mathbf{Y} and \mathbf{A} specifies the starting point of the constitutive law of the present thermodynamic formulation as

$$\dot{\mathbf{Y}}(\mathbf{y}, \mathbf{z}) = \mathbf{a}''(\mathbf{y}, \mathbf{z}) \cdot \dot{\mathbf{y}} + \mathbf{b}(\mathbf{y}, \mathbf{z}) \cdot \dot{\mathbf{z}}, \quad -\dot{\mathbf{A}}(\mathbf{y}, \mathbf{z}) = \mathbf{b}^T(\mathbf{y}, \mathbf{z}) \cdot \dot{\mathbf{y}} + \mathbf{g}(\mathbf{y}, \mathbf{z}) \cdot \dot{\mathbf{z}} \tag{5}$$

with $\mathbf{a}''(\mathbf{y}, \mathbf{z}) = E_{,yy}$ being the Tisza matrix and $\mathbf{b}(\mathbf{y}, \mathbf{z}) = E_{,zy}$, $\mathbf{g}(\mathbf{y}, \mathbf{z}) = E_{,zz}$ the coupling and the dissipation matrices, respectively, [21].

The thermodynamic information as summarized by the state laws given previously has to be completed by the evolution laws of the internal variables, which in turn have to be incorporated into the previous Lagrangian. In the more general setting of the theory of irreversible process (abbreviated T.I.P.) or the thermodynamics with internal variables (abbreviated T.I.V.), one has recourse to the notion of a dissipation potential D , constrained to be a positive and homogeneous function of degree n in the rate of the internal variables, [19], allowing to express the thermodynamic forces or affinities \mathbf{A} versus the dual internal variable \mathbf{z} as

$$\mathbf{A}(\mathbf{z}) = \frac{\partial D}{\partial \dot{\mathbf{z}}}. \tag{6}$$

In the previous relation and in the sequel, one assumes that the thermodynamic affinity includes all dissipative forces. Using the previous relation results in a dissipation expressed as

$$\Phi = \mathbf{A} \cdot \dot{\mathbf{z}} = \frac{\partial D(\mathbf{z})}{\partial \dot{\mathbf{z}}} = nD \geq 0, \tag{7}$$

by virtue of Euler’s identity for homogeneous functions of degree n (in classical T.I.P., D is taken as homogeneous of degree two). The previous relation can be inverted to give the rate of the internal variable versus the affinity, using the Legendre–Fenchel transform, involving the pseudo-potential of dissipation, defined as the conjugate

$$D^*(\mathbf{A}) = \sup_{\mathbf{A}} (\mathbf{A} \cdot \dot{\mathbf{z}} - D(\dot{\mathbf{z}})), \tag{8}$$

where \mathbf{A} is restricted to a convex set K , [19]. When $D^*(\mathbf{A})$ is differentiable, the evolution law of the internal variable \mathbf{z} is given by

$$\dot{\mathbf{z}} = \frac{\partial D^*(\mathbf{A})}{\partial \mathbf{A}}. \tag{9}$$

Choosing as a specific model a quadratic and convex pseudo potential of dissipation D^* , one obtains the following set of independent nonlinear kinetic relations (for a number of N processes)

$$\dot{z}_k = -\frac{z_k - z_k^r}{\tau_k}, \quad k = 1, \dots, N, \tag{10}$$

considering a spectral decomposition of the dissipative phenomena modelled by internal variables \dot{z}_k . The variables z_k^r represent the value of the internal variable z at its relaxed equilibrium state, for the dissipation mode k ; the relaxed state corresponds to a metastable equilibrium state for which the thermodynamic affinity A^r vanishes. The distribution of the varying relaxation times therein τ_k is described from the Prigogine fluctuation theorem, with the contribution of each mode proportional to the square root of the corresponding relaxation time [21]. Such first-order kinetic equations are classical in various fields of engineering, with a good illustration in chemistry, whereby they constitute models for chemical reactions linking the reaction rate with concentrations or pressures of reactants (only one reactant is involved in a first-order rate equation) and constant parameters, such as rate coefficients.

2.2 Lagrangian formulation of the set of constitutive laws (Phase DM-P1)

A Lagrangian formulation of the previous dissipative constitutive laws is further constructed by the homotopy formula from the self-adjoint system of PDEs (partial differential equations) (5), accounting for the Gibbs relation and the Gibbs–Duhem relations (4); see [22,23]:

$$L = E_{,y} \cdot \dot{y} + E_{,z} \cdot \dot{z} = \frac{dE(y, z)}{dt}. \tag{11}$$

This Lagrangian will be coined the thermodynamic Lagrangian below, since it incorporates the thermodynamic information related to the material, in terms of relations between the extensive control variables and the dual intensive thermodynamic forces, where the generalized coordinates $\{y, z, \dot{y}, \dot{z}\}$ correspond to the Lagrange variables of the uniform system. The physical meaning of the obtained result is that the set of internal variables completes the set of control variables, ensuring the self-adjointness of the constitutive equations, as detailed in [22]. Indeed, the Maxwell relations for the internal energy potential, viz.

$$E_{,y_i y_j} = E_{,y_j y_i}; \quad E_{,y_i z_j} = E_{,z_j y_i}; \quad E_{,z_i z_j} = E_{,z_j z_i} \tag{12}$$

imply that the first-order differential form dE evaluated from (2) is exact, i.e., that the integral of dE between two thermodynamic states (y_1, z_1) and (y_2, z_2) of the material at times t_1 and t_2 , respectively, is independent of the path joining them, thus

$$\delta E = \delta \left(\int_{(y_1, z_1)}^{(y_2, z_2)} dE \right) = \delta \int_{t_1}^{t_2} \frac{dE}{dt} dt = 0 \tag{13}$$

with δ the functional variation. Note that for a different set of control variables, one has to use the adequate potential (the material time derivative of which being identified with the Lagrangian), constructed as the Legendre–Fenchel transform of the internal energy. The kinetic information, viz. the set of equations (10) is then incorporated into the previous thermodynamic Lagrangian via Lagrange multipliers $\lambda = \{\lambda_k, k = 1, \dots, N\}$; hence the augmented Lagrangian, L_{aug} , accounting for both the thermodynamic and the kinetic information inherent in the material’s constitutive law, can be written as

$$l_{\text{aug}} = \dot{e} + \sum_{k=1}^N \lambda_k \left(\dot{z}_k + \frac{z_k - z_k^r}{\tau_k} \right), \tag{14}$$

where the internal energy E has been replaced by the internal energy density e , which also satisfies the Maxwell conditions. The Lagrangian density l_{aug} in Eq. 14 is the volumetric density of a Lagrangian L_{aug} , such that

$$L_{\text{aug}} = \int_V l_{\text{aug}} dV. \tag{15}$$

The Lagrangian multipliers in l_{aug} are akin to thermodynamic affinities dual to the rate of internal variables; hence the kinetic part of the total Lagrangian thereabove resembles a dissipation potential. The incorporation of the kinetic laws (10) clearly breaks the symmetry (discrete symmetry) under the time reversal $t \rightarrow -t$, since the new Lagrangian contains irreversible information. As necessary conditions, the set of Lagrangian equations is completely equivalent to the constitutive equations (5) and the kinetic equations (10). The kinetic evolution equations (9) have a rich content in terms of continuous symmetries, according to the specific form taken by the relaxation time therein.

2.3 Symmetry analysis of the augmented Lagrangian (Phase DM-P2)

Relying on the formulated least-action principle of the dissipative constitutive laws, we next exploit the associated variational symmetries to determine invariance properties of the set of constitutive equations, thereby relying on the methodology presented in [22–25]. Those symmetries depend upon the thermodynamic and kinetic informations encapsulated into the two thermodynamic potentials entering the Lagrangian (the energy density and the dissipation potential). We focus in the sequel (but without loss of generality) on the specific energy $e(\varepsilon, s, z_k)$, depending on the strain ε , the specific entropy s , and some specific internal variables z_k , $k = 1, \dots, N$, accounting for internal dissipative phenomena (related to the microstructure).

The search for the variational symmetries of the Jacobi action built from the augmented Lagrangian amounts to finding the infinitesimal generators on the (first-order) jet space sustained by the variables $\{t, \varepsilon, s, z_k\}$. As a concrete illustration, and without loss of generality, considering the time t as the sole independent variable and as the dependent variables the vector $\mathbf{y} = \varepsilon, s, z_k$, we express a generator of a symmetry group as

$$\mathbf{v}_{\text{cont}} = \xi \frac{\partial}{\partial t} + \phi^\varepsilon \frac{\partial}{\partial \varepsilon} + \phi^s \frac{\partial}{\partial s} + \phi^{z_k} \frac{\partial}{\partial z_k}, \quad (16)$$

where the variations of the dependent variables are given by

$$\delta \varepsilon = \mu \phi^\varepsilon; \quad \delta s = \mu \phi^s; \quad \delta z_k = \mu \phi^{z_k}, \quad (17)$$

with μ the parameter of the group, accounting for the formalism presented in [5]. In Eq. 16, the subscript “cont” means that only components with respect to control variables are at first introduced. The variations given in (17) are responsible for the variations of the intensive observable variables $\delta \sigma$, δT , and δA_k , obtained from the second partial derivatives of the potential e . The symmetry group acting on the set of variables $\{t, \varepsilon, s, z_k\}$ is automatically extended to the enlarged set of variables $\{t, \varepsilon, s, z_k, \sigma, T, A_k\}$: thus, the total vector field, generator of the symmetries of the constitutive laws, is decomposed into the sum of the “control” vector field and the “observable” vector field:

$$\mathbf{v} = \mathbf{v}_{\text{cont}} + \mathbf{v}_{\text{obs}} \quad \text{with} \quad \mathbf{v}_{\text{obs}} = \phi^\sigma \frac{\partial}{\partial \sigma} + \phi^T \frac{\partial}{\partial T} + \phi^{A_k} \frac{\partial}{\partial A_k}. \quad (18)$$

The components of the intensive variables of the observable and internal vector fields are given by the structure of the constitutive law (Eq. 5) for the elementary representative volume element (RVE):

$$\phi^\sigma = e_{,\varepsilon\varepsilon} \phi^\varepsilon + e_{,s\varepsilon} \phi^s + e_{,z_k\varepsilon} \phi^{z_k}, \quad (19)$$

$$\phi^T = e_{,\varepsilon s} \phi^\varepsilon + e_{,s s} \phi^s + e_{,z_k s} \phi^{z_k}, \quad (20)$$

$$\phi^{A_i} = e_{,\varepsilon z_i} \phi^\varepsilon + e_{,s z_i} \phi^s + e_{,z_k z_i} \phi^{z_k}. \quad (21)$$

The components of the vector field \mathbf{v}_{obs} are thus completely determined by those of the vector field \mathbf{v}_{cont} . Let us now return to a general situation, and decompose the Lagrangian of Eq. 14 into a thermodynamic and a kinetic contribution, as

$$l_{\text{aug}} = l_{\text{thermo}} + l_{\text{kine}} \quad (22)$$

with

$$l_{\text{thermo}} = \dot{e} \quad \text{and} \quad l_{\text{kine}} = \sum_{k=1}^N \lambda_k \left(\dot{z}_k + \frac{z_k - z_k^r}{\tau_k} \right). \tag{23}$$

As shown previously, the invariance condition associated with the sole contribution l_{thermo} is automatically verified; see also [22]. The satisfaction of the Euler–Lagrange equations for \dot{e} as a Lagrangian is fully equivalent to the Maxwell conditions for the second-order partial derivatives of the potential function e . Thus, the invariance condition of the action integral simplifies to the differential condition involving the sole kinetic information (and kinetic Lagrangian)

$$\text{pr}^{(1)}\mathbf{v}l_{\text{kine}} + l_{\text{kine}}\text{Div}\boldsymbol{\xi} = 0, \tag{24}$$

accounting for the relation $\text{Div}\boldsymbol{\xi} = D_t\boldsymbol{\xi} = \dot{\boldsymbol{\xi}}$. In Eq. 24, $\text{pr}^{(1)}\mathbf{v}$ denotes the first-order prolongation of the vector field \mathbf{v} ; see e.g. [5].

The connection between the variational symmetry condition and the local symmetry of the field equations is established next. If \mathbf{v} is a variational symmetry for some functional $S = \int_t \int_V l dV dt$, it is also a symmetry of the corresponding Lagrange equations:

$$\text{pr}^{(1)}\mathbf{v}l + l\text{Div}\boldsymbol{\xi} = 0 \quad \Rightarrow \quad \text{pr}^{(1)}\mathbf{v}(E(l)) = 0, \tag{25}$$

but the converse is generally not true: the set of all variational symmetries denoted by G_S is always included into the set of local symmetries, G_Δ , i.e., $G_S \subset G_\Delta$. In the present case, we only consider variational symmetries along the optimal path where the constitutive laws are satisfied. Accounting for the expression of l_{kine} , viz.

$$l_{\text{kine}} = \sum_{k=1}^N \lambda_k \left(\dot{z}_k + \frac{z_k - z_k^r}{\tau_k} \right), \tag{26}$$

we satisfy the condition $l_{\text{kine}} = 0$ along the optimal path; hence the variational symmetry condition becomes

$$\text{pr}^{(1)}\mathbf{v}l_{\text{kine}} = 0 \quad \text{whenever} \quad l_{\text{kine}} = 0. \tag{27}$$

Thanks to the linearity of the prolongation, the previous condition becomes

$$\sum_{k=1}^N \lambda_k \left(\text{pr}^{(1)}\mathbf{v} \left(\dot{z}_k + \frac{z_k - z_k^r}{\tau_k} \right) \right) = 0 \quad \text{whenever} \quad \sum_{k=1}^N \lambda_k \left(\dot{z}_k + \frac{z_k - z_k^r}{\tau_k} \right) = 0. \tag{28}$$

Doing a spectral decomposition of the dissipative mechanisms, the evolutions of the internal variables are mutually independent, hence the symmetry conditions (28) may be decoupled, resulting in a new set of N independent symmetry conditions:

$$\text{pr}^{(1)}\mathbf{v} \left(\dot{z}_k + \frac{z_k - z_k^r}{\tau_k} \right) = 0 \quad \text{whenever} \quad \dot{z}_k + \frac{z_k - z_k^r}{\tau_k} = 0, \tag{29}$$

which is equivalent to the local symmetry condition. The same condition holds true when using the more general form of the kinetic equations according to the previously established framework; see the evolution equation (9). This allows extending the symmetry analysis to more complex dissipative phenomena.

The interest of this formulation in terms of symmetries is linked to the pseudo-potential of dissipation, viz. the function $D^*(\mathbf{A})$. The vector field selected as the infinitesimal generator of the general invariance condition (27) is also split into the sum of a control and an observable vector field, with the components of the last contribution completely expressed from the components of the control vector; this amounts to searching for symmetries in a subset of the total jet space (with the state laws for the observable variables being automatically satisfied). A classification of symmetries according to the form of this function is one perspective of the present contribution. As a partial conclusion, the results of the previous developments show that variational symmetries along the optimal path are fully equivalent to local symmetries directly computed from the constitutive laws.

2.4 Invariance properties of the constitutive equations (Phase DM-P2)

The constitutive law for dissipative materials is considered as given in the direct problem; its formulation relies on the well-established thermodynamics of irreversible processes, involving potential functions for the writing of the state laws. In this section, the importance of the continuous symmetries inherent in those dissipative constitutive laws is outlined, especially in view of setting up a predictive methodology allowing to condense the material's response into equivalence principles. From the latter, it may be possible to define an equivalent experimental set-up allowing a gain of time by selecting an optimal set of control variables (e.g. temperature, strain rate). Considering, for instance, a general internal-variable formulation of inelasticity for generalized standard materials, some of those symmetries are sometimes being implicitly reflected into the form taken by two essential scalar-valued functions: the thermodynamic potential (or the pseudo potential of dissipation introduced in (8)) and the dissipation function. Recall that such a general thermodynamically based material framework has its roots in [26–29]; it has proven its ability to cover a broad spectrum of models in viscoelasticity, viscoplasticity, plasticity, and also continuum damage mechanics.

The methodology presented in the previous sections, based on the search for the Lie groups of constitutive laws, is further used as a systematic tool for the construction of the so-called master curves that condense the information related to the behavior of a material under varying experimental conditions. Those master curves are a practical confirmation (in the language of the engineer) of the invariance properties resulting from the symmetry analysis, and are in line with the representation of experimental data gained from the various measurements done on different materials with the purpose to validate the calculated symmetries. It is important to underline that experiments enter into play only in the very last phase of the Direct Method, as a validation of the calculated symmetry groups.

The experimental conditions are defined by the values taken by a set of control parameters such as temperature, strain rate. Hence, the knowledge of symmetry groups allows a prediction regarding the modification of the material's response when these parameters vary. Thus, starting from a known set of constitutive equations, i.e., a known expression for e , z_k^r and τ_k , it is a priori possible to compute some symmetries of the behavior by applying the symmetry condition (29). Due to the equivalence of the local and variational symmetry conditions, a conservation obtained using Noether's theorem, viz. $\text{Div } \mathbf{P} = 0$, with

$$P_i = \sum_{k=1}^q \sum_{j=1}^4 \xi_j u_{k,j} \frac{\partial l_{\text{kin}}}{\partial u_{k,i}} - \sum_{j=1}^q \phi_j \frac{\partial l_{\text{kin}}}{\partial u_{j,i}} - \xi_i l_{\text{kin}}, \quad (30)$$

see e.g. [5], with l_{kin} the kinetic Lagrangian density, defined in the general thermodynamic setting exposed in the previous section as

$$l_{\text{kin}} = \sum_k \lambda_k \left(\dot{z}_k - \frac{\partial D^*(A)}{\partial A_k} \right) \equiv 0. \quad (31)$$

Recall that the arguments of the dependent variables of the kinetic Lagrangian, viz. the variables u_k , are internal variables. For a uniform elementary representative volume element (over which the kinetic Lagrangian density is being integrated), only the time derivative appears in the previous conservation-law expression.

The search for generalized conservation laws associated with dissipative balance equations is an aspect under current development, that will not be considered in the present contribution.

2.5 Application: construction of the master curve of a dry polyamid (PA66) (Phase DM-P3 → DM-P5)

The direct problem listed in the introductory part of this paper involves the calculation of the Lie symmetries for the BVP on a uniform RVE, for which the constitutive law is the field equation to be analyzed. The corresponding invariants will further be translated into master curves, as explained below. The constitutive model set-up presently for the purpose of uniaxial tests involves the strain ϵ as control variable, and the stress σ as the corresponding observable variable. The temperature T , dual of the entropy s , plays the role of a parameter. The internal variables

z_k and the thermodynamic affinities A_k are not controlled; their values in the relaxed state are governed by the strain history. The exploitation of the general constitutive equations (5) and (10) leads to the following state law (see [30]):

$$\begin{aligned} \dot{\sigma} - E_u \dot{\varepsilon} + \sum_{k=1}^n b_k^1 \frac{z_k - z_k^r}{\tau_k} &= 0, & -\dot{s} + \alpha_u E_u \dot{\varepsilon} + \sum_{k=1}^n b_k^2 \frac{z_k - z_k^r}{\tau_k} &= 0, \\ -\dot{A}_i - b_i^1 \dot{\varepsilon} + \sum_{k=1}^n g_{ik} \frac{z_k - z_k^r}{\tau_k} &= 0, & i = 1, \dots, n \end{aligned} \tag{32}$$

with E_u the constant instantaneous Young’s modulus, b_k^1 , α_u (dilatation coefficients), b_k^2 , and g_{ik} being defined from the second-order partial derivatives of the Helmholtz free energy; see [31]. The simplest evolution model for the internal variables is

$$z_k^r = c_k \varepsilon. \tag{33}$$

If one starts from a particular expression of the kinetic laws, involving a temperature dependence of the relaxation times of the Arrhenius kind, the following kinetic model may be written as

$$\dot{z}_k + \frac{z_k - c_k \varepsilon}{\frac{h}{kT} \exp\left(\frac{\Delta H - T \Delta S_k}{RT}\right)} = 0, \tag{34}$$

where h is the Planck constant, k the Boltzmann constant, R the gas constant, c_k , ΔH , ΔS_k some material constants. The required symmetry condition can accordingly be expressed as

$$\text{pr}^{(1)} \mathbf{v} \left(\dot{z}_k + \frac{z_k - c_k \varepsilon}{\frac{h}{kT} \exp\left(\frac{\Delta H - T \Delta S_k}{RT}\right)} \right) = 0. \tag{35}$$

The particular solution is obtained as

$$\mathbf{v}_0 = \xi \frac{\partial}{\partial t} + \phi^T \frac{\partial}{\partial T} + \phi^\sigma \frac{\partial}{\partial \sigma} + \phi^s \frac{\partial}{\partial s} + \phi^{A_k} \frac{\partial}{\partial A_k} \tag{36}$$

with:

$$\xi = t; \quad \phi^T = -\frac{RT^2}{RT + \Delta H}; \quad \phi^\sigma = -\alpha_u E_u \phi^T; \quad \phi^{A_k} = -b_k^2; \quad \phi^T; \quad \phi^\varepsilon = 0; \quad \phi^{z_k} = 0. \tag{37}$$

This solution is further interpreted as a mathematical formulation of the so-called time–temperature equivalence principle for polymers. The integration of the induced first-order differential system leads to a one-parameter group of transformations, where the temperature is transformed as

$$\bar{T} = \exp\left(L_W \left(\frac{\Delta H}{R} \exp\left(\frac{\mu T - T \log(T^*) + \frac{\Delta H}{R}}{T} \right) \right) - \mu + \log T^* - \frac{\Delta H}{RT} \right) \tag{38}$$

with $L_W(x)$ the Lambert function, and $T^* = T/T_0$ with $T_0 = 1$ K. The theoretical shift factor is obtained by inverting the previous implicit relation:

$$\frac{\mu(T, \bar{T})}{\log 10} = \frac{\Delta H(T - \bar{T})}{RT \bar{T} \log 10} + \frac{1}{\log 10} \log \frac{T}{\bar{T}}. \tag{39}$$

This explicit expression of the group parameter $\mu(T, \bar{T})$ highlights an invariance property satisfied by the secant modulus $E_s(t, T)$, which is defined as the ratio of the stress to the strain:

$$E_s(t, T) = \frac{\sigma(t, T)}{\varepsilon(t, T)}. \tag{40}$$

Indeed, it can be readily shown that:

$$E_s(t, T) = \frac{\sigma(t, T)}{\varepsilon(t, T)} = \frac{\bar{\sigma}(\bar{t}, \bar{T})}{\bar{\varepsilon}(\bar{t}, \bar{T})} = E_s(\bar{t}, \bar{T}) = E_s(e^{\mu} t, \bar{T}) \tag{41}$$

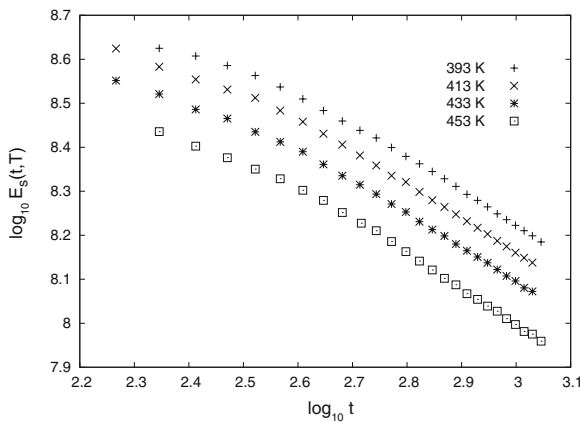


Fig. 2 Experimental secant modulus $E_s(t, T)$ versus time t ($\log_{10} - \log_{10}$ representation) for PA66 at different temperatures, from [31]

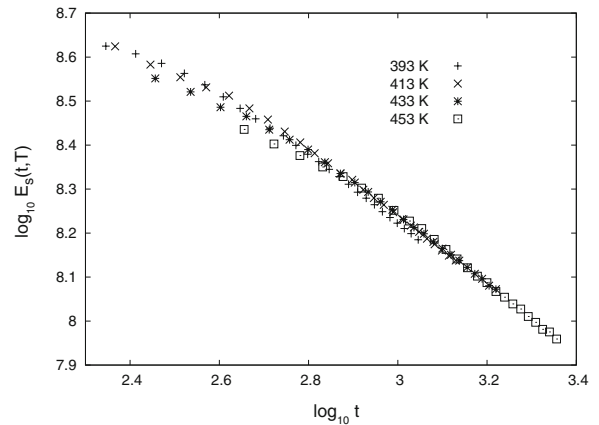


Fig. 3 Master curve obtained at 393 K by translation of curves of Fig. 2 along the time axis, from [31]

leading to the logarithmic expression

$$\log_{10} \bar{t} = \log_{10} t + \frac{\mu}{\log 10} \tag{42}$$

allowing to rewrite (41) as:

$$E_s(\log_{10} t, T) = E_s\left(\log_{10} t + \frac{\mu}{\log 10}, \bar{T}\right). \tag{43}$$

This relation links the two secant modulus $E_s(\log_{10} \bar{t}, \bar{T})$ and $E_s(\log_{10} t, T)$ obtained at \bar{T} and T by a translation of factor $\frac{\mu}{\log 10}$ on the time logarithmic scale. This translation factor $\frac{\mu}{\log 10}$ is given explicitly in Eq. 39. In fact, the obtained invariance property is nothing but a theoretical formulation of the time–temperature equivalence principle (abbreviated as TTEP). The latter was successfully confirmed by experimental data on various materials, and fitted with empiric relationships such as the Williams–Landel–Ferry (WLF) expression (see e.g. [32–34]) or the Kohlrausch relation ([35–37]).

As an illustration of the TTEP, and referring to the experimental validation of the predicted shift factor (DM-P5), let us consider the data summarized in Fig. 2. The evolution of the secant modulus $E_s(t, T)$ for isothermal tests on a polymer (polyamid 66, PA66) is plotted at different constant temperatures and for a given strain rate ($\dot{\epsilon} = 1.8 \times 10^{-4} \text{ s}^{-1}$). Every curve is parameterized by the temperature \bar{T} (in the range 413–453 K) and can be translated along the $\log_{10} t$ axis with a horizontal shift factor $a_{\bar{T} \rightarrow T}$ to coincide with a unique curve. This unique curve is called “master curve” at the reference temperature $T = 393 \text{ K}$; see Fig. 3.

This property has been stated by Williams, Landel and Ferry as the following empirical rule:

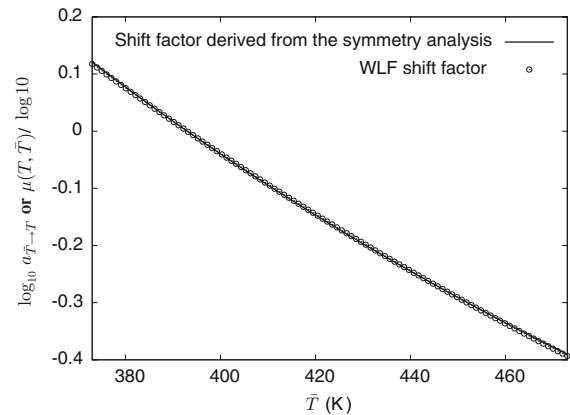
$$E_s(\log_{10} t, T) = E_s(\log_{10} t + \log_{10} a_{\bar{T} \rightarrow T}, \bar{T}), \tag{44}$$

which is nothing but the reformulation of (43) with $\log_{10} a_{\bar{T} \rightarrow T} \equiv \frac{\mu}{\log 10}$. The expression of the WLF shift factor is

$$\log_{10} a_{\bar{T}, T} = -\frac{C_1(\bar{T} - T)}{C_2 + \bar{T} - T} \tag{45}$$

with C_1 and C_2 some material constants. The comparison between the WLF and the predicted shift factor (Eqs. 45, and 39) for the dry polyamid (PA66) given in Fig. 4 shows a good agreement between the proposed predictive method and the empirical WLF approach.

Fig. 4 Comparison between the shift factor expressed by WLF empirical model and the present model, from [31]



It can be noticed that the Direct Method exposed in this section may be extended to more complicated constitutive laws including complex nonlinearities. This is the object of the next section.

2.6 Towards an extension of the Direct Method

The previous methodology (DM) has clearly shown its efficiency for predicting and formulating master curves, providing a set of constitutive equations. Nevertheless, the algorithm may be generalized to a *class* of constitutive equations. More precisely, if a general class of behaviors is considered, that is, if they are written in terms of generic functions, the search of the associated Lie groups allows a classification of symmetries depending on the mathematical structure of these functions. This may result in (i) a synthetic calculation of master curves belonging to the same “family”, and (ii) the prediction of master curves in a broader constitutive framework (e.g. nonlinear elasticity, viscoplasticity, etc). The algorithm of this Extended Direct Method, synthesized in the diagram of Fig. 5, has been recently applied to the case of nonlinear elastodynamics in [14] and, in the same manner, to the case of non-local elasticity in [38].

For a given system of PDEs, a symmetry classification can be done with respect to point symmetries or non-local symmetries with generators depending on nonlocal variables. Those nonlocal variables—also coined potential variables—arise from conservation laws admitted by a given system of differential equations: each conservation law introduces one auxiliary potential variable, which is nonlocally related to the original dependent variables; see [39–42] in the context of wave equations. Extension of this method to the construction of a hierarchy of nonlocally related systems can be achieved by introducing new potential variables, one for each new conservation law of each potential system. Furthermore, one may obtain nonlocally related subsystems by excluding dependent variables from the potential system. This allows the construction of a whole tree of nonlocally related potential systems and subsystems originating from a given system of DEs [41]. Finding trees of potential systems is important when arbitrary constitutive functions are present in the original set of differential equations. In the present context, those functions govern the motion, strain and rheological properties of some class of materials. For different forms of the constitutive functions, the sets of conservation laws and subsequent trees of related potential systems and subsystems may be different.

As already pointed out, one can systematically calculate nonlocal symmetries according to the constitutive model, allowing the construction of invariants (new conservation laws can be obtained) and new solutions (linearization of the set of original DEs is even sometimes possible). Moreover, any general method of analysis for a given set of DEs may be tried on nonlocally related potential systems or subsystems, since those subsystems include all solutions of the original system.

As an illustration of those concepts and methods, nonlocally related PDE systems for one-dimensional nonlinear elastodynamics have been formulated in [14], and point symmetries of PDE systems of dynamical nonlinear elasticity have been classified with respect to constitutive and loading functions (body force). For physically realistic

constitutive functions, new closed-form dynamical solutions have been constructed from symmetry reduction of the PDE system under nonlocal symmetries, viz. symmetries involving nonlocal variables. In order to illustrate those ideas, let us consider a one-dimensional situation; the Euler equations of motion read (x and t are the independent variables):

$$\mathbf{E}\{x, t; v, \sigma, \rho\} = 0 : \begin{cases} \rho_t + (\rho v)_x = 0, \\ \sigma_x + \rho f(x, t) = \rho(v_t + vv_x), \\ \sigma = K(\rho), \end{cases} \quad (46)$$

where ρ , v , σ are the density, velocity and Cauchy stress, respectively. The third relation in (46) represents the constitutive law involving an arbitrary function $K(\rho)$. The body force $f(x, t)$ is considered as an arbitrary loading function, in accordance with the Phase EDM-P1 (see Fig. 5). The conservation of mass—first relation in Eq. 46—introduces a potential variable w ; hence a new nonlocally related PDE system can be constructed:

$$\mathbf{EW}\{x, t; v, \sigma, \rho, w\} = 0 : \begin{cases} w_x = \rho, \\ w_t = -\rho v, \\ \sigma_x + \rho f(x) = \rho(v_t + vv_x), \\ \sigma = K(\rho). \end{cases} \quad (47)$$

It is clear from the last system that w is nonlocally related to the mass and the velocity. An integration is thus needed for its evaluation.

Considering materials endowed with dissipation, a symmetry classification of constitutive laws belonging to a broader class (viscoelastic, viscoplastic) is a promising perspective of the Extended Direct Method. Indeed, the general form of the constitutive law involves arbitrary functions of the strain, strain rate and a set of internal variables related to dissipation. As the complexity of the set of governing DEs is increased—due to the nonlinearity of the constitutive law in the considered broader class—the use of potential variables would simplify the finding of solutions and invariance relations (Phase EDM-P3).

In the next section, the inverse problem of symmetry analysis listed in the introductory section will be studied.

3 The inverse method: determination of the material's constitutive law from experimental data and Lie symmetries

3.1 The IM algorithm

As discussed in the introductory section, the inverse problem involves the construction of a possible constitutive law for the material. This method is based on a search of symmetry groups coming from experimental data of suitable mechanical testings. More precisely, the algorithm consists in the following consecutive steps:

- Obtain empirically master curves from the whole set of experimental data, and express these master curves in terms of Lie groups of generators $\mathbf{v}_i^{\text{exp}}$, $i = 1, \dots, n$ (Phases IM-P1 to IM-P3);
- Choose a general expression for the constitutive equation, formally expressed as a relationship between (for instance) the Cauchy stress, the Henky strain and the associated strain rate, e.g.:

$$\Delta(\sigma, \varepsilon, \dot{\varepsilon}) = \sigma - f(\varepsilon, \dot{\varepsilon}) = 0, \quad (48)$$

where $f(\varepsilon, \dot{\varepsilon})$ is an unknown function;

- Apply the symmetry condition ([5]):

$$\mathbf{v}_i^{\text{exp}} \Delta = 0 \quad \text{whenever } \Delta = 0 \quad (49)$$

for all generators $\mathbf{v}_i^{\text{exp}}$ found in IM-P3, in order to specify the form of the general function $f(\varepsilon, \dot{\varepsilon})$ (Phase IM-P4 to Phase IM-P6). From a physical viewpoint, Eq. 49 means that the same constitutive law holds for all considered values of the (varying) loading parameters. Otherwise stated, one looks for a model of the constitutive laws

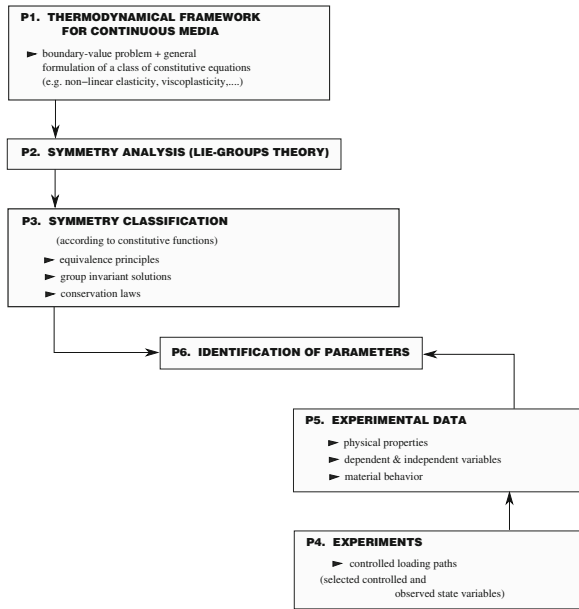


Fig. 5 Diagram associated with the Extended Direct Method

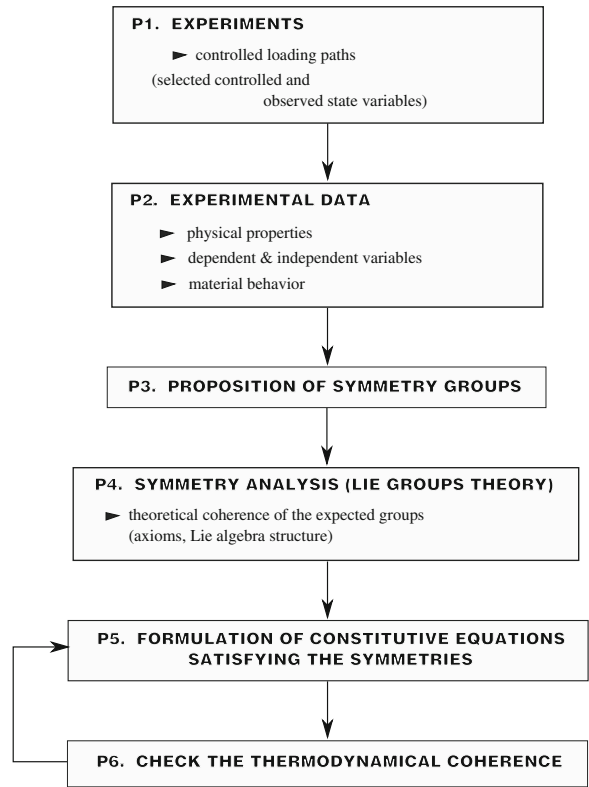


Fig. 6 Diagram associated with the Inverse Method

that satisfies the symmetries obtained from the experimental responses. The constructed equations have to be admissible from a thermodynamical point of view; hence this gives an additional criterium to be checked (IM-P5 and IM-P6 as a loop).

Those steps are condensed into the diagram of Fig. 6.

As an illustration of this algorithm, let us consider the case of a stick-like material. The constitutive equations are defined as unknown functions of mechanical variables such as the stress, strain and strain rate. They are fully determined considering the invariance of the constitutive model with respect to the identified symmetry groups.

3.2 Application: formulation of a Lie group for acrylic sticks from experimental data (Phases IM-P1 → IM-P4)

The cylindrical acrylic stick specimens are submitted to compressive dynamic tests, in accordance with the experimental device depicted in Fig. 7. A unit mass (1 kg) is dropped with no initial velocity from a height H_0 to impact the upper plate of a compression device onto contact with the specimen.

The displacement $\delta(t)$ of the upper plate (relative to the lower plate) and the applied load are measured during impact; hence the Cauchy stress $\sigma(t)$, the Henky strain $\varepsilon(t)$, and the strain rate $\dot{\varepsilon}(t)$ can be evaluated from those data using the following relations:

$$\sigma(t) = \frac{F(t)h(t)}{S_0h_0}; \quad \varepsilon(t) = \log \frac{h(t)}{h_0}; \quad \dot{\varepsilon} = \frac{\dot{h}(t)}{h(t)} \tag{50}$$

with $h(t) = h_0 - \delta(t)$ the actual specimen thickness, assuming that the total volume of the specimen is constant during the compression ($V(t) = S(t)h(t) = S(t = 0)h(t = 0) = S_0h_0$). This is a consequence of the local

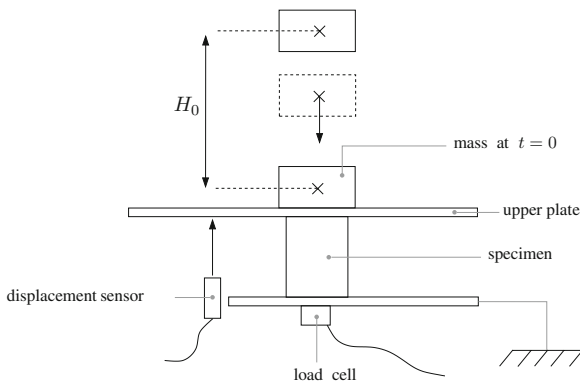


Fig. 7 Schematic representation of the weight drop tests, from [30]

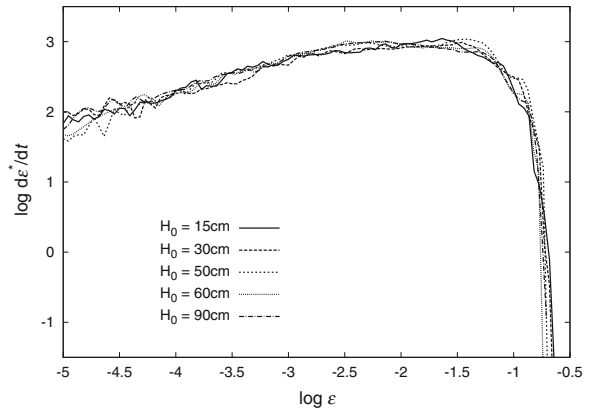


Fig. 8 Master curve obtained in the representation $(\log \epsilon, \log \dot{\epsilon}^*)$, from [30]

incompressibility, which is a reasonable assumption for such polymers. Five series of dynamic compressive tests have been carried out with this protocol, considering initial heights from 15 to 90 cm. Since the impact velocity increases with the drop height, this amounts to considering different strain rates.

The experimental data obtained in the phase plane $(\epsilon, \dot{\epsilon})$ and in the strain–stress plane (ϵ, σ) for the five values of H_0 may be condensed in both planes into two master curves by means of suitable transformations. Those master curves reveal a superposition of the measured sets of data obtained for different values of the drop height, which is the varying parameter H_0 . For $H_0 = 90$ cm as a reference height, shift factors $a_{x \rightarrow 90}$, $x = 15, 30, 50, 60$ such that

$$\begin{cases} \epsilon_{90}(t) = a_{x \rightarrow 90} \epsilon_x(t) \\ \dot{\epsilon}_{90}(t) = a_{x \rightarrow 90} \dot{\epsilon}_x(t) \end{cases} \quad (51)$$

have been graphically determined by a translation along the vector $(1, 1)$ in the logarithmic plane $(\log \epsilon, \log \dot{\epsilon}^*)$, with $\dot{\epsilon}^*$ the dimensionless strain rate; see Fig. 8. Similarly, the Cauchy stress versus Henky strain responses have been synthesized into master responses involving the affine transformation

$$\log \left(\frac{\sigma_{90}^*}{\sigma_0^*} \right) = b_{x \rightarrow 90} \log \left(\frac{\sigma_x^*}{\sigma_0^*} \right), \quad (52)$$

where the five constant $b_{x \rightarrow 90}$ for $x = 15, 30, 50, 60$, and σ_0^* have also been graphically determined. The master curve revealing the stress–strain responses when the drop height varies is shown in Fig. 9.

The values of the measured shift factors $a_{x \rightarrow 90}$ and $b_{x \rightarrow 90}$ for $x = 15, 30, 50, 60$ are given in Tables 1 and 2 respectively, defining the additional trivial values $a_{90 \rightarrow 90} = 1$ and $b_{90 \rightarrow 90} = 1$; they are to be used to obtain the formulation of a Lie group. Considering a one-parameter group of transformations, an exponential adjustment of the shift factors $b_{x \rightarrow 90}$ versus $\log a_{x \rightarrow 90}$ is found suitable:

$$b_{x \rightarrow 90} = \beta e^{\alpha \log a_{x \rightarrow 90}} \quad \text{with } \beta = 1.045, \alpha = 1.687, R^2 = 0.9924. \quad (53)$$

Table 1 Values of shift factors $\log a_{x \rightarrow 90}$, from [30]

Parameter	$\log a_{15 \rightarrow 90}$	$\log a_{30 \rightarrow 90}$	$\log a_{50 \rightarrow 90}$	$\log a_{60 \rightarrow 90}$	$\log a_{90 \rightarrow 90}$
Value	0.52	0.36	0.36	0.22	0

Table 2 Values of parameters $b_{x \rightarrow 90}$ and σ_0^* , from [30]

Parameter	$b_{15 \rightarrow 90}$	$b_{30 \rightarrow 90}$	$b_{50 \rightarrow 90}$	$b_{60 \rightarrow 90}$	$b_{90 \rightarrow 90}$	σ_0^* (Pa)
Value	2.48	1.97	1.53	1.53	1	59,900

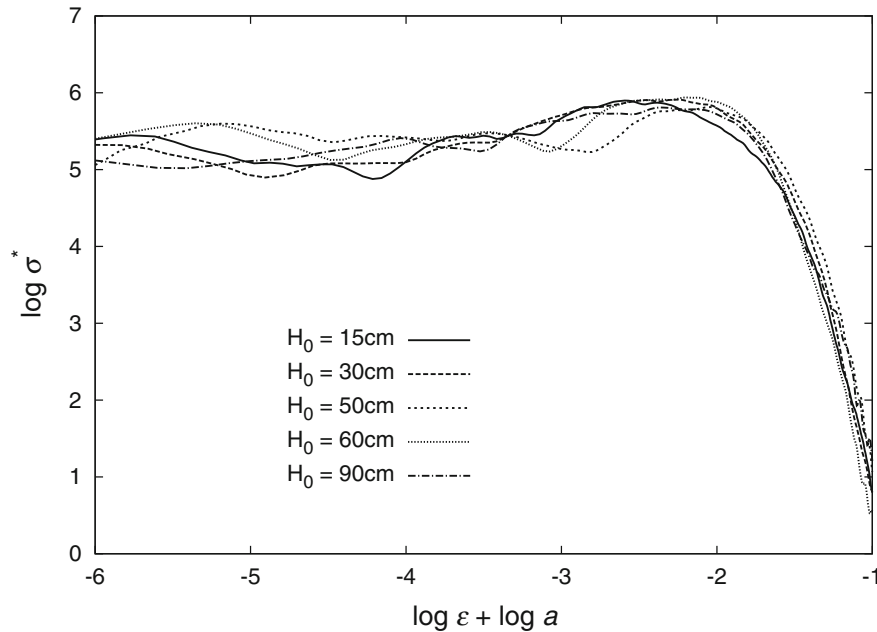


Fig. 9 Master curve obtained in the representation $(\log \varepsilon + \log a, \log \sigma^*)$, from [30]

Finally, denoting $\mu_x = \log a_{x \rightarrow 90}$ and considering Eq. 51 and the exponential of (52), the Lie group associated with experiments at varying heights writes:

$$\bar{\varepsilon} = e^\mu \varepsilon \tag{54}$$

$$\bar{\dot{\varepsilon}}^* = e^\mu \dot{\varepsilon}^* \tag{55}$$

$$\frac{\bar{\sigma}^*}{\sigma_0^*} = \left(\frac{\sigma^*}{\sigma_0^*} \right)^{\beta e^{\alpha \mu}} \tag{56}$$

This group expresses the correspondence between $(\varepsilon, \dot{\varepsilon}^*, \sigma^*)$ and $(\bar{\varepsilon}, \bar{\dot{\varepsilon}}^*, \bar{\sigma}^*)$ for two different values of H_0 .

3.3 Formulation of the constitutive equations (Phase IM-P5)

Using the obtained Lie group, one further evaluates the constitutive law from the previous symmetry condition (49) applied to Eq. 48, accounting for the generator of the group, viz:

$$\mathbf{v}^{\text{exp}} = \varepsilon \frac{\partial}{\partial \varepsilon} + \dot{\varepsilon}^* \frac{\partial}{\partial \dot{\varepsilon}^*} + \alpha \sigma^* \log \left(\frac{\sigma^*}{\sigma_0^*} \right) \frac{\partial}{\partial \sigma^*} \tag{57}$$

The final form of the constitutive relation between the stress, the strain and strain rate reads

$$\sigma = \sigma_0 \left(c_1 \left(\frac{\varepsilon}{\dot{\varepsilon}^*} \right)^{c_2} \right)^{(\dot{\varepsilon}^*)^\alpha} \tag{58}$$

with $\sigma_0 = \sigma_0^* \times 1 \text{ Pa}$ and $\alpha = 1.687$, see [30] for the complete calculation. The material parameters c_1, c_2 are estimated from the adjustment of the formulated constitutive model with the measurements, in a least-squares sense:

$$c_1 = 1.05, \quad c_2 = 0.053. \tag{59}$$

The range of validity of the developed model has to be explored with further experimental data obtained at both lower and higher strain rates than those used in the present study. Predictions of the constitutive model will then be compared with those measurements, in order to assess the relevance of the extrapolation of the proposed constitutive

law to a wider range of strain rates. The power of the modelling strategy presented here lies in its ability to account for a set of experimental data in a synthesized manner. Indeed, all experimental data are summarized in a Lie group, which allows to efficiently determine the form of the constitutive equation.

4 Concluding remarks and perspectives

Lie symmetries can be used as a powerful tool in Mechanics of Materials, as highlighted by the previous developments and examples. First it was shown (in the Direct Method section) that a Lagrangian relying on a given kinetic information is able to bring out variational symmetries. Accounting for the expression of the associated pseudo-potential, such a symmetry analysis may be a promising tool to predict the existence of master curves. The method has next been extended to some class of constitutive equations, written in terms of unknown functions describing the loading path or the material behavior. This generalized algorithm (called Extended Direct Method) consists of a classification of symmetries with respect to the unknown functions. It results in an efficient calculation of master curves that can be carried out in a broader constitutive framework.

In the second part of the paper, it has been evidenced that Lie symmetries may be used as an interpolation method, called here the Inverse Method. In the first step of the algorithm, one looks for one or several Lie symmetries formulated from experimental master curves. Next, the subsequent symmetry conditions are applied to a general expression of the constitutive equations involving some unknown functions. This results in a PDE system satisfied by these unknown functions. The solution of the system and the confrontation of the obtained constitutive equations with experimental data give an idea on the relevance of the model.

Both previous analyses (Direct and Inverse Methods) will be carried out in a more systematic manner in the future, in particular for nonlinear and dissipative constitutive models (e.g. plasticity, viscoplasticity of materials). Furthermore, work in the background of the Inverse Method is in progress, with the aim of identifying constitutive laws for a wide range of materials under various loading conditions. Complementary to that, invariants of the associated BVPs (i.e., conservation laws) can be calculated in the framework of the Extended Direct Method. In nonlinear Continuum Mechanics, those invariants may be used to develop suitable numerical schemes that preserve the invariance condition along the solution path.

References

1. Ibragimov NH (1994) CRC Handbook of Lie group analysis of differential equations. Symmetries, exact solutions, and conservation laws, 2nd edn, vol 2. CRC Press, Boca Raton, FL
2. Ibragimov NH (1995) CRC Handbook of Lie group analysis of differential equations. Applications in engineering and physical sciences, 2nd edn, vol 2. CRC Press, Boca Raton, FL
3. Ibragimov NH (1996) CRC Handbook of Lie group analysis of differential equations. New trends in theoretical developments and computational methods, 2nd edn, vol 2. CRC Press, Boca Raton, FL
4. Bluman GW, Anco SC (2002) Symmetry and integration methods for differential equations. Springer, New York
5. Olver P (1993) Applications of Lie groups to differential equations. Springer, New York
6. Ovsianikov LV (1982) Group analysis of differential equations. Academic Press, New York
7. Bluman GW, Kumei S (1989) Symmetries and differential equations. Springer, New York
8. Dorodnitsyn V, Winternitz P (2000) Lie point symmetry preserving discretizations for variable coefficient Korteweg-de Vries equations. *Modern group analysis. Nonlinear Dyn* 22:49–59
9. Ozer T (2003) Symmetry group classification for one dimensional elastodynamics problems in nonlocal elasticity. *Mech Res Commun* 30(6):539–546
10. Chien N, Honein T, Hermann G (1996) Dissipative systems, conservation laws and symmetries. *Int J Solids Struct* 33(20–22):2959–2968
11. Chien N, Honein T, Hermann G (1993) Conservation laws for linear viscoelasticity. *Zeit Angew Math Phys* 44:44–52
12. Senashov SI (1980) Invariant spatial solutions for equations of ideal plasticity. *Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki* 2:139–142
13. Annin BD, Bytev VO, Senashov SI (1985) Group properties of equations of elasticity and plasticity. Nauka, Novosibirsk
14. Bluman GW, Cheviakov AF, Ganghoffer JF (2008) Nonlocally related pde systems for one-dimensional nonlinear elastodynamics. *J Eng Math* 62(3):203–221

15. Budianski B, Rice JR (1973) Conservation laws and energy-release rate. *ASME J Appl Mech* 40:201–203
16. Maugin GA (1993) *Material inhomogeneities in elasticity*. Cambridge University Press, Cambridge
17. Steinmann P, Maugin GA (2005) *Mechanics of material forces*, volume 11 of *Advances in mechanics and mathematics*. Springer, New York
18. Eshelby JD (2006) *The mechanics of defects and inhomogeneities - collected works of J. D. Eshelby*, volume 133 of *Solid mechanics and its applications*. Springer
19. Maugin GA (1999) *The thermodynamics of nonlinear irreversible behaviours. An introduction*. World Scientific, Singapore
20. Callen HB (1985) *Thermodynamics: an introduction to the physical theories of equilibrium thermostatics and irreversible thermodynamics*, 2nd edn. Wiley, New York
21. Cunat C (2001) The DNLR approach and relaxation phenomena. Part I: historical account and DNLR formalism. *Mech Time-Dependant Mater* 5(1):39–65
22. Magnenet V, Rahouadj R, Ganghoffer JF, Cunat C (2007) Continuous symmetries and constitutive laws of thermo-elasto(visco) plastic materials within a thermodynamic framework of relaxation. Part I: formal aspects. *Int J Plast* (23):87–113
23. Rahouadj R, Ganghoffer JF, Cunat C (2003) A thermodynamic approach with internal variables using Lagrange formalism. Part I: general framework. *Mech Res Commun* 30(2):109–117
24. Rahouadj R, Ganghoffer JF, Cunat C (2003) A thermodynamic approach with internal variables using Lagrange formalism. Part II: Continuous symmetries in the case of the time-temperature equivalence. *Mech Res Commun* 30(2):119–123
25. Magnenet V, Rahouadj R, Ganghoffer JF, Cunat C (2004) Master curves for viscous media predicted from Lie symmetry analysis. In: *Proceedings of the symposium on trends in applications of mathematics to mechanics*. Darmstadt, Germany, August
26. Biot MA (1956) *Variational and Lagrangian methods in viscoelasticity*. Springer, Berlin
27. Biot MA (1955) Variational principles in irreversible thermodynamics with application to viscoelasticity. *Phys Rev* 97(6):1463–1469
28. Ziegler H (1963) Some extremum principles in irreversible thermodynamics with applications to continuum mechanics. In: Sneddon R, Hill IN (eds) *Progress in solid mechanics*, vol IV. North-Holland, Amsterdam
29. Germain P (1998) Functional concepts in continuum mechanics. *Meccanica* 33(5):433–444
30. Magnenet V, Rahouadj R, Ganghoffer JF (2008) A new methodology for determining the mechanical behavior of polymers exploiting Lie symmetries: application to a stick-like material. *Mech Mater* (in press)
31. Magnenet V, Rahouadj R, Ganghoffer JF, Cunat C (2009) Continuous symmetry analysis of a dissipative constitutive law: application to the time-temperature superposition. *Eur J Mech A* 28:744–751
32. Hotta A, Terentjev EM (2001) Long-time stress relaxation in polyacrylate nematic liquid crystalline elastomers. *J Phys* 13:11453–11464
33. Laot C (2001) Gas transport properties in polycarbonate. PhD thesis, Virginia Polytechnic Institute
34. Ricco T, Pegoretti A (2000) Nonlinear dynamic behavior of rubber compounds: construction of dynamic moduli generalized master curves. *Polym Eng Sci* 40:2227–2231
35. Meyer A, Busch R, Schober H (1999) Time-temperature superposition of structural relaxation in a viscous metallic liquid. *Phys Rev Lett* 83:5027–5029
36. O'Connell PA, McKenna GB (1997) Large deformation response of polycarbonate: time temperature, time aging time and time strain superposition. *Polym Eng Sci* 37:1442–1448
37. Nicholson LM, Whistley KS, Gates TS (2001) Crosslink density and molecular weight effects on the viscoelastic response of a glassy high-performance polyimide. *Polymer (Preprints)* 42
38. Ozer T (2008) An application of symmetry groups to nonlocal continuum mechanics. *Comput Math Appl* 55:1923–1942
39. Bluman G, Kumei S (1988) On invariance properties of the wave equation. *J Math Phys* 28:307–318
40. Bluman G, Kumei S, Reid G (1988) New classes of symmetries for partial differential equations. *J Math Phys* 29:806–811
41. Bluman G (1993) Use and construction of potential symmetries. *Math Comput Model* 18(10):1–14
42. Cheviakov A, Bluman G (2007) Nonlocally related system, linearization and nonlocal symmetries for the nonlinear wave equation. *J Math Anal Appl* 333:93–111